Second assignment SIO212C, 2019

Hand-in for the lecture on the 14th of February

1 Some homework problems

The horizontal deformation flow

(i) Find the solution of the 2d advection equation

$$c_t - \alpha x c_x + \alpha y c_y = 0\,, \tag{1}$$

with the initial condition

$$c(x, y, 0) = c_* \tanh\left(\frac{x\,\sin\chi - y\,\cos\chi}{\ell_0}\right)\,. \tag{2}$$

The initial condition is a front of thickness ℓ_0 tilted at an angle χ to the x-axis. *(ii)* Describe the solution in twenty or thirty words: how does the orientation of the front change with time? Discuss the limit as $t \to \infty$. *(iii)* Suppose now that $\kappa \neq 0$, so that

$$c_t - \alpha x c_x + \alpha y c_y = \kappa(c_{xx} + c_{yy})\,. \tag{3}$$

The initial condition has $\ell_0 \gg \ell_\kappa \stackrel{\text{def}}{=} \sqrt{\kappa/\alpha}$. Discuss the effects of non-zero κ: when does κ first become important, and how is $\kappa = 0$ solution in part *(ii)* altered? In considering this question pay attention to the effects of χ: how does t_κ depend depend on χ? *(iv)* Assuming that $\chi \neq 0$, and making some plausible assumptions, find the ultimate $t \to \infty$ steady solution of the advection-diffusion equation with the initial condition above.

Optional: the method of moments

Use the method of moments to solve

$$c_t + \beta y c_x = \kappa(c_{xx} + c_{yy})\,, \tag{4}$$

with the Gaussian initial condition

$$c_0(x, y) = \frac{A}{2\pi\ell_0^2} \exp\left(-\frac{x^2 + y^2}{2\ell_0^2}\right)\,. \tag{5}$$

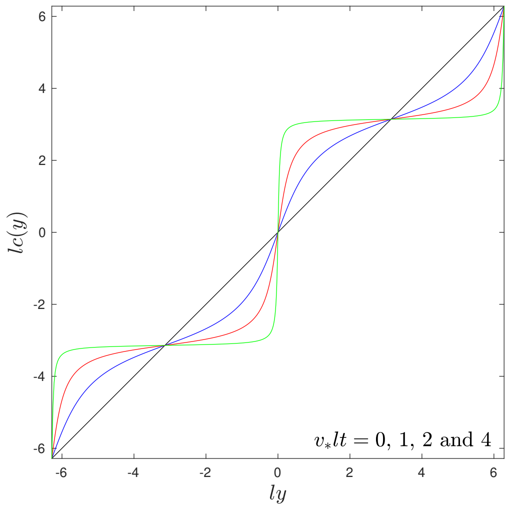

Figure 1: Solution of Stone's problem.

Optional: Front formation by sharpening an initially uniform gradient

Solve the advection equation

$$c_t + J(v_* x \sin ly, c) = 0\,, \qquad \text{with initial condition} \qquad c(x,y,0) = y\,. \tag{6}$$

Plot the solution at selected times: see figure 1. This example from Stone (1966) shows a smooth passive-scale ramp being sharpened into a staircase by a simple incompressible velocity field.

Notation: $J(\ ,\ )$ denotes the "Jacobian":

$$J(\psi, c) = \psi_x c_y - \psi_y c_x = u c_x + v c_y\,. \tag{7}$$

2 References

1. Stone, P. H. (1966). Frontogenesis by horizontal wind deformation fields. *Journal of the Atmospheric Sciences*, **23**, 455-465.

2. Williams, R. T., & Plotkin, J. (1968). Quasi-geostrophic frontogenesis. *Journal of the Atmospheric Sciences*, **25**, 201-206.

3. Hoskins, B. J. (1982). The mathematical theory of frontogenesis. *Annual Review of Fluid Mechanics*, **14**, 131-151.

4. Hoskins, B. J., Draghici, I., & Davies, H. C. (1978). A new look at the ω-equation. *Quarterly Journal of the Royal Meteorological Society*, **104, 31-38.**