First assignment SIO212C, 2019

Hand-in for the lecture on Thursday 17th

The Boussinesq Equations

Here are the Boussinessq equations

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \hat{\boldsymbol{z}} \times f\boldsymbol{u} + \boldsymbol{\nabla} p = b\hat{\boldsymbol{z}}\,, \tag{1}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} = 0\,, \tag{2}$$

$$\boldsymbol{\nabla\cdot u} = 0\,, \tag{3}$$

where $\mathrm{D}/\mathrm{D}t \stackrel{\text{def}}{=} \partial_t + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}$ is the convective derivative.

(i) Identify four or five approximations to the exact equations of fluid mechanics that have been made in getting to (1) through (3). *(ii)* How is the buoyancy, b, defined in terms of ρ, g and so on? What are the dimensions of b and p? *(iii)* Consider a ΔT of 1°C in the mixed layer. What is the order of magnitude of the corresponding Δb? *(iv)* Write the equation for the vorticity,

$$\boldsymbol{\omega} \stackrel{\text{def}}{=} \boldsymbol{\nabla}\times\boldsymbol{u}\,, \tag{4}$$

in a form that is pleasing to you. Does Kelvin's circulation theorem apply to the Boussinesq equations? *(v)* Complete the following results

$$\frac{\mathrm{D}\frac{1}{2}|\boldsymbol{u}|^2}{\mathrm{D}t} + \boldsymbol{\nabla\cdot F} = ??? \tag{5}$$

$$\frac{\mathrm{D}(-zb)}{\mathrm{D}t} = ??? \tag{6}$$

Give expressions for the flux $\boldsymbol{F}$ and identify the terms indicated by ??? on the right. Interpret these results in physical terms. *(vi)* Show that

$$\frac{\mathrm{D}\Pi}{\mathrm{D}t} = 0\,, \tag{7}$$

where the potential vorticity is

$$\Pi \stackrel{\text{def}}{=} (\boldsymbol{\omega} + f\hat{\boldsymbol{z}})\boldsymbol{\cdot}\boldsymbol{\nabla} b\,. \tag{8}$$

(While it is not strictly necessary, you'll find it easy to prove material conservation of Π if you first complete the kinematic problem below.)

A kinematic result

Suppose $c(\boldsymbol{x},t)$ is a passive scalar and $\boldsymbol{\xi}(\boldsymbol{x},t)$ is a passive vector, both advected by an incompressible velocity $\boldsymbol{u}$:

$$\frac{\mathrm{D}c}{\mathrm{D}t} = 0\,, \qquad \text{and} \qquad \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} = \boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}\,. \tag{9}$$

Show that

$$\frac{\mathrm{D}(\boldsymbol{\nabla} c\boldsymbol{\cdot}\boldsymbol{\xi})}{\mathrm{D}t} = 0\,. \tag{10}$$

The Kolmogorov scale in a coffee cup

Suppose you stir $10^{-4}\mathrm{m}^3$ of coffee and you observe that on the scale of the cup (i.e., the largest possible length scale) the eddy turnover velocity is about $0.1\mathrm{m\ s}^{-1}$. Estimate the rate of mechanical energy dissipation, ε, and thus the Kolmogorov scale in the coffee cup. (Work with "logarithmic" or "ballpark" accuracy.)

Given that the molecular diffusivity of large fat molecules in water is about $10^{-12}\mathrm{m}^2\mathrm{s}^{-1}$, estimate Batchelor scale for cream. (We've been reading the *Journal of Agricultural and Food Chemistry* looking for this number — $10^{-12}\mathrm{m}^2\mathrm{s}^{-1}$ is a guess. Sugar is around $0.5 \times 10^{-9}\mathrm{m}^2\mathrm{s}^{-1}$.)